Does Church-Turing thesis apply outside computer science?

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Abstract

We analyze whether Church-Turing thesis can be applied to mathematical and physical systems. We find the factors that allow to a class of systems to reach a Turing or a super-Turing computational power. We illustrate our general statements by some more concrete theorems on hybrid and stochastic systems.

The future of mathematics comes from informatics, the future of informatics comes from mathematics

Thoughts, Jean Della Dora

1 Introduction

The discovery in the first half of the XX-th century of several computational models (such as Post and Turing machines, recursive functions, Markov's normal algorithms etc.), all capable to realize any imaginable algorithm, and the proof of their equivalence, can be seen as the first step of the computer science. They determined the invention of computers, design of programming languages, and the development of a new kind of mathematics that evolved to theoretical computer science. All the above-mentioned computational models are based on unbounded discrete-time, unbounded discrete-state memory, deterministic programs, and no-noise no-faults execution. Such restrictions were a key to the correct (and unique) definition of computability, as well as to the digital computers, based on special engineering tricks allowing to represent discrete state, discrete time, and noise rejection in the physical world.

According to Church-Turing thesis, any reasonable (i.e. satisfying above-mentioned restrictions) computational model leads to the same (or smaller) class of computable functions as Turing machines, and hence Turing machines capture the general notion of algorithm.

However a challenging research direction consists in considering any natural class of dynamical systems (possibly continuous in time and/or state-space, possibly non-deterministic, possibly noisy, possibly quantic) as a computational model, and exploring its computational power. It could be interesting for several reasons. The most important one is to verify or to falsify the following thesis.

Thesis 1.1 (Extended Church-Turing). *Feasible, realizable, reproducible physical computing devices have the same (or smaller) computational power as Turing machines.*

In other words, according to this thesis, there is no way to build a computer, based on some new principles and capable to "compute" other functions than Turing computable ones. As far as I know, there is no scientific evidence in favor of this thesis, but almost everybody believes in it. Since it is difficult to define rigorously what a feasible *physical* device is, a possible approach to this thesis could be to consider various classes of *mathematical* models, to analyze their computational power, and whenever it exceeds the power of Turing machines, to think about its physical feasibility.

Another scientific application of the computability approach to dynamical systems is to use the computational power as a natural measure of behavioral complexity. The bigger is the set of functions computed by a class of dynamical systems, the more complex, strange, pathologic can be the behavior of such systems. In fact complexity hierarchies of the theoretical computer science (similar to forgotten hierarchies of the descriptive set theory, but easier to study) are quite rich, and can shed new light on chaos and other strange attractors. A very high level of complexity can be interpreted as a sign of unrealizability of the system.

Last but not least, practically speaking the computational approach to dynamical systems is a key to the decidability analysis of verification problems concerning hybrid and continuous systems.

The computability view of dynamical systems is not really new, it is related to research on computability on reals[12], super-Turing models, hybrid systems, cellular automata and so on.

In this talk, several concrete research works on various aspects of computability by dynamical systems will be presented. The author was involved in these works during last 12 years, see [1]-[5].

2 Continuous systems as computational models

Most of results will be established for a class of piecewise-constant differential equations (so-called PCD systems) that appears in Hybrid systems research, and its variants. Computability by such devices will be defined, and their computational power will be explored in following situations.

Dynamical systems weaker then TMs : Poincaré - Bendixson's paradise[1, 4]. It is well-known that differential equations on the plane have rather simple global behavior. In the computational paradigm this leads a weak (sub-Turing) computational power of planar systems.

Dynamical systems as strong as TMs : chaos[1]. Starting from three dimensions dynamical systems admit chaotical behaviors. We will analyze how the chaos allows to match the full Turing computational power (see also [10]).

Far beyond Turing : Zeno's horror[2, 6]. The so-called Zeno phenomenon for piecewise continuous (and even piecewise constant) systems consists in the simple fact that a trajectory can have infinitely many changes of the type of dynamics during a finite interval of time. Such a behavior is much more complex than the simple deterministic chaos. This kind of extremely complex trajectories can be used to decide any arithmetic predicate in a bounded lapse of time. Hence the computational power of Zeno dynamical systems goes far beyond Turing machines. We will speculate about physical (non)-realizability of Zeno computers because of their extreme sensibility to perturbations.

Small set-valued noise: upside down TMs[3]. Several researchers (in particular Henzinger, Raskin [9] and Fränzle[7]) suggested to consider more realistic, imprecise, perturbed dynamical systems. A folk conjecture suggests that adding a small imprecision to the dynamics makes the reachability problem decidable, by destroying too subtle behaviors, and replacing them by thick "tubes" that rapidly cover parts of the state-space. However, up to now there are no convincing decidability results in this direction.

Inspired by Puri's work [11] we have tried another approach: replace the exact dynamics of a system by an ϵ-perturbed contingent one with $\epsilon \to 0$. As the result the computational power becomes Π_1^0. This means that instead of recognizing recursively enumerable sets, "perturbed" systems recognize complements of such sets. We will explain the reason of this inversion in computational power.

Large deviations or small stochastic noise: one step beyond Turing[5]. Another one, maybe more realistic model takes into account a small stochastic noise, and to pass to the limit as it is often done in the large deviations research (see [8]). A characterization of the computational power of such perturbed systems will be presented. It turns out to be Δ_2^0, i.e. slightly superior to Turing machines. We will relate it with the complexity of behaviors appearing as large deviations of deterministic dynamics.

3 Conclusions

We believe that the computability approach to dynamical systems can be a source of new insights and new research problems. We will present some important open problems that exist in this area.

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